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2009 J. Phys. A: Math. Theor. 42 105203

(http://iopscience.iop.org/1751-8121/42/10/105203)

J. Phys. A: Math. Theor. 42 (2009) 105203 (12pp) doi:10.1088/1751-8113/42/10/105203

On bi-Hamiltonian geometry of some integrable systems on the sphere with cubic integral of motion

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Received 21 November 2008, in final form 19 January 2009
Published 11 February 2009
Online at stacks.iop.org/JPhysA/42/105203

Abstract
We obtain the bi-Hamiltonian structure for a family of integrable systems on the sphere $\mathcal{S}^2$ with an additional integral of third order in momenta. These results are applied to the Goryachev system and Goryachev–Chaplygin top for which we give an explicit procedure to find the separated coordinates and the separated relations.

PACS numbers: 45.10.Na, 45.40.Cc
Mathematics Subject Classification: 70H20, 70H06, 37K10

1. Introduction

We address the problem of the separation of variables for the Hamilton–Jacobi equation within the theoretical scheme of bi-Hamiltonian geometry. We want to learn to calculate the bi-Hamiltonian structure for a given integrable system on the Poisson manifold M with the Poisson bivector P and the Casimir functions C_a. The separation variables are naturally associated with this bi-Hamiltonian structure of M itself [6].

According to [12–14] we will suppose that desired second Poisson bivector is the Lie derivative of P along some unknown Liouville vector field X on M

$$P' = \mathcal{L}_X(P) \tag{1.1}$$

which has to satisfy to the equation

$$[P', P'] = [\mathcal{L}_X(P), \mathcal{L}_X(P)] = 0, \qquad \Leftrightarrow \qquad \left[\mathcal{L}^2_X(P), P\right] = 0 \tag{1.2}$$

with respect to the Schouten bracket [., .]. By definition (1.1) bivector P' is compatible with a given bivector P, i.e. $[P, P'] = 0$. In geometry such bivector P' is said to be the two-coboundary associated with the Liouville vector field X in the Poisson–Lichnerowicz cohomology defined by P on M.

From all the solutions X of equation (1.2) we have to choose partial solutions X such that

$$\{H_i, H_j\}' = 0, \qquad i, j = 1, \ldots, n, \tag{1.3}$$

where H_j are the given integrals of motion and $\{.,.\}'$ is the Poisson bracket associated with the Poisson bivector P' (1.1).

Obviously enough, in their full generality the system of equations (1.2), (1.3) is too difficult to be solved because it has infinitely many solutions labeled by different separated coordinates, see [14]. In order to get particular solutions we will use addition assumptions

$$P' dC_a = 0, \qquad a = 1, \dots, k, \tag{1.4}$$

and some special ansätze for the Liouville vector field X.

Equations (1.4) say that we are looking for the Poisson bivector P', which has the same foliation by symplectic leaves as P. In general bivector P' could have some more Casimirs, so that their symplectic leaf could be smaller, but in the separation of variables method we have to consider equivalent foliations only [12–14].

2. A family of integrable systems on the sphere

A Hamiltonian system is called natural if its Hamiltonian is the sum of a positive-definite kinetic energy and a potential. Natural Hamiltonian systems on cotangent bundles of closed surfaces admitting integrals polynomial in momenta are especially interesting [3]. In this section we define some family of natural Hamiltonian systems on $T^*\mathcal{S}^2$ with the quadratic and cubic in the momenta integrals of motion.

Let two vectors $J = (J_1, J_2, J_3)$ and $x = (x_1, x_2, x_3)$ are coordinates on the Euclidean algebra $e^*(3)$ with the Lie-Poisson bracket

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \qquad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \qquad \{x_i, x_j\} = 0, \tag{2.1}$$

where ε_{ijk} is the totally skew-symmetric tensor. This bracket has two Casimir functions

$$C_1 = |x|^2 \equiv \sum_{k=1}^{3} x_k^2, \qquad C_2 = (x, J) \equiv \sum_{k=1}^{3} x_k J_k. \tag{2.2}$$

Fixing their values one gets a generic symplectic leaf of $e^*(3)$

$$\mathcal{O}_{AB}: \qquad \{x, J : C_1 = A, C_2 = B\},$$

which is a four-dimensional symplectic manifold.

At $C_2 = 0$ this symplectic manifold is equivalent to cotangent bundle $T^*\mathcal{S}^2$ of the sphere

$$\mathcal{S}^2 = \{x \in \mathbb{R}^3, |x| = A\}.$$

For the Liouville integrability of the equations of motion it is enough to find one additional integral of motion H_2, which is functionally independent of the Hamiltonian H_1 and the Casimir functions.

In this $e^*(3)$ coordinates the family of integrable systems on the sphere at $C_2 = 0$ is defined by the following quadratic Hamiltonian:

$$H_1 = J_1^2 + J_2^2 + (3\alpha^2 + f(x_3))J_3^2 + m(x_3)x_1 + g(x_3), \tag{2.3}$$

and cubic additional integral of motion

$$H_2 = -2\alpha J_3\big(-\alpha^2 J_3^2 + J_1^2 + J_2^2 + f(x_3)J_3^2 + g(x_3)\big) - n(x_3)J_1 - \ell(x_3)x_1 J_3. \tag{2.4}$$

Here

$$\begin{gathered} g(x_3) = \frac{a^2 b}{n(x_3)^2}, \qquad m(x_3) = -\frac{n'(x_3)}{\alpha}, \qquad \ell(x_3) = \frac{n(x_3)n''(x_3)}{n'(x_3)}, \\ f(x_3) = 1 - 3\alpha^2 - \alpha\frac{3x_3 m(x_3) - 2\big(A^2 - x_3^2\big)m'(x_3)}{n(x_3)} + \frac{x_3\ell(x_3) - \big(A^2 - x_3^2\big)\ell'(x_3)}{n(x_3)}, \end{gathered} \tag{2.5}$$

and

$$n(x_3) = -a(x_3 + c)^{\beta}, \qquad a, b, c \in \mathbb{R}. \tag{2.6}$$

At $C_2 = 0$ integrals of motion H_1 (2.3) and H_2 (2.4) are in the involution with respect to the brackets (2.1) in the following five cases:

$$\begin{array}{llll} 1. & \alpha = \beta = 1, & c = 0, & n(x_3) = -ax_3, \\ 2. & \alpha = \beta = \frac{1}{3}, & c = 0, & n(x_3) = -ax_3^{1/3}, \\ 3. & \alpha = \beta = \frac{1}{6}, & c = A, & n(x_3) = -a(x_3 + A)^{1/6}, \\ 4. & \alpha = \beta = \frac{1}{2}, & c \in \mathbb{R}, & n(x_3) = -a(x_3 + c)^{1/2}, \\ 5. & \alpha = \beta = \frac{1}{4}, & c = A, & n(x_3) = -a(x_3 + A)^{1/4}. \end{array} \tag{2.7}$$

The corresponding Hamiltonian (2.3) depends on parameters a, b and c

$$H_1^{(1)} = J_1^2 + J_2^2 + 4J_3^2 + ax_1 + \frac{b}{x_3^2},$$

$$H_1^{(2)} = J_1^2 + J_2^2 + \frac{4}{3}J_3^2 + \frac{ax_1}{x_3^{2/3}} + \frac{b}{x_3^{2/3}},$$

$$H_1^{(3)} = J_1^2 + J_2^2 + \left(\frac{7}{12} + \frac{x_3}{2(x_3 + A)}\right) J_3^2 + \frac{ax_1}{(x_3 + A)^{5/6}} + \frac{b}{(x_3 + A)^{1/3}},$$

$$H_1^{(4)} = J_1^2 + J_2^2 + \left(1 + \frac{x_3}{x_3 + c} - \frac{x_3^2 - c^2}{4(x_3 + c)^2}\right) J_3^2 + \frac{ax_1}{(x_3 + c)^{1/2}} + \frac{b}{x_3 + c},$$

$$H_1^{(5)} = J_1^2 + J_2^2 + \left(\frac{13}{16} + \frac{3x_3}{8(x_3 + A)}\right) J_3^2 + \frac{ax_1}{(x_3 + A)^{3/4}} + \frac{b}{(x_3 + A)^{1/2}}.$$

The Hamilton function $H_1^{(1)}$ describes the well-studied generalized Goryachev–Chaplygin top [1]. In contrast with this case the other systems have no physical meaning, however they may be interesting as some mathematical toys. The second integrable system with the Hamiltonian $H_1^{(2)}$ was found by Goryachev [2]. The Hamilton function $H_1^{(4)}$ was studied by Dullin and Matveev [4]. The third and fifth integrable systems with Hamiltonians $H_1^{(3)}$ and $H_1^{(5)}$ have been found in [9].

3. Construction of the Liouville vector field

In this section we describe how to obtain the desired Liouville vector field X and the corresponding second Poisson bivector P' for our family of integrable systems. Remind that smooth manifold M endowed with a pair of compatible Poisson bivectors P and P' is said to be ωN manifold if one of the Poisson brackets is non-degenerate [6].

Without loss of generality we can always put $|x| = A = 1$ by using scaling transformation $x_j \to A^{-1}x_j$ and consider the unit sphere $\mathcal{S}^2$ only. On its cotangent bundle $T^*\mathcal{S}^2$ we introduce the following coordinates:

$$\phi = \arctan\left(\frac{x_1}{x_2}\right), \qquad u = x_3, \qquad p_\phi = -J_3, \qquad p_u = \frac{J_1x_2 - x_1J_2}{x_1^2 + x_2^2},$$

so that

$$J_1 = \frac{u}{\sqrt{1-u^2}}\sin(\phi)p_\phi + \sqrt{1-u^2}\cos(\phi)p_u, \qquad x_1 = \sqrt{1-u^2}\sin(\phi),$$

$$J_2 = \frac{u}{\sqrt{1-u^2}}\cos(\phi)p_\phi - \sqrt{1-u^2}\sin(\phi)p_u, \qquad x_2 = \sqrt{1-u^2}\cos(\phi),$$

$$J_3 = -p_\phi, \qquad x_3 = u.$$

In these coordinates u, ϕ, p_u, p_ϕ initial Poisson tensor P associated with the brackets (2.1) becomes canonical tensor on a two-dimensional symplectic manifold

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{3.1}$$

which is non-degenerate.

Calculations in these coordinates are faster than calculations in the standard spherical coordinates $\phi, \theta = \arccos u$. Of course, using these coordinates on symplectic leaf of $e^*(3)$ we implicitly accepted assumptions (1.4) for the initial Poisson manifold $e^*(3)$ with the Casimir functions (2.2). More precisely, there are some different extensions of the Poisson tensors on symplectic leaf of $e^*(3)$ to the whole $e^*(3)$ and one of them satisfies to equations (1.4).

The Hamiltonian (2.3) in these coordinates looks like

$$\begin{aligned} H_1 &= (1-u^2)p_u^2 + \left(3\alpha^2 + f(u) + \frac{u^2}{1-u^2}\right) p_\phi^2 + \sqrt{1-u^2}\sin(\phi) m(u) + g(u) \\ &= \mathrm{g}_u p_u^2 + \mathrm{g}_\phi p_\phi^2 + V(u,\phi), \end{aligned} \tag{3.2}$$

where

$$\mathrm{g}_u = (1-u^2), \qquad \mathrm{g}_\phi = 3\alpha^2 + f(u) + \frac{u^2}{1-u^2}.$$

According to [10] in order to solve equations (1.2), (1.3) we will use polynomial in momenta *ansätze* for the components of the Liouville vector field $X = \sum X^j \partial_j$:

$$X^j = \sum_{k=0}^{N} \sum_{m=0}^{k} y^j_{km}(u,\phi) p_u^{k-m} p_\phi^m. \tag{3.3}$$

For all the systems we put $N = 2$, it means that X^j will be generic second-order polynomials in momenta p_u, p_ϕ with coefficients $y^j_{km}(u,\phi)$ depending on variables u and ϕ.

Substituting this ansatz (2.6) into the equations (1.2)–(1.3) and demanding that all the coefficients at powers of p_u and p_ϕ vanish one gets the over determined system of 60 algebro-differential equations on the 24 functions $y^j_{km}(u,\phi)$ which can be easily solved in the modern computer algebra systems.

Below we discuss all the obtained solutions.

3.1. Particular solution

For the Goryachev–Chaplygin top [1] with integral of motion $H^{(1)}_{1,2}$ one gets the following:

Proposition 1. *At the first case $\alpha = \beta = 1$ and $n(x_3) = ax_3$ there is one particular solution depending on coordinate u only*

$$P' = \begin{pmatrix} 0 & u & \frac{up_\phi}{1-u^2} & (1-u^2)p_u \\ * & 0 & \frac{(1-2u^2)p_u}{1-u^2} & \frac{(2-u^2)p_\phi}{1-u^2} \\ * & * & 0 & up_u^2 + \frac{up_\phi^2}{1-u^2} + \frac{b}{u^3} \\ * & * & * & 0 \end{pmatrix}. \tag{3.4}$$

The entries of P' are real functions on initial variables.

By construction this Poisson bivector compatible with P (3.1) and functions $H_{1,2}$ are in bi-involution with respect to the corresponding Poisson brackets. So that the phase space $T^*\mathcal{S}^2$ becomes a semisimple ωN manifold and the foliation defined by $H_{1,2}$ is separable in the Darboux–Nijenhuis variables [6].

The variables of separation $q_{1,2}$ (the Darboux–Nijenhuis variables) are the eigenvalues of the recursion operator $N = P'P^{-1}$. They are simply roots of the following minimal characteristic polynomial of N

$$\mathcal{A}(\lambda) = (\det(N-\lambda\mathrm{I}))^{1/2} = \lambda^2 - 2p_\phi\lambda - (1-u^2)p_u^2 - \frac{u^2p_\phi^2}{1-u^2} - \frac{b}{u^2}$$
$$= \lambda^2 + 2J_3\lambda - J_1^2 - J_2^2 - \frac{b}{x_3^2}. \tag{3.5}$$

At $b = 0$ these variables of separation have been found by Chaplygin in [1]. In initial $e^*(3)$ variables this bivector

$$P' = \begin{pmatrix} 0 & -x_3^2 & x_3x_2 & -x_2J_1 & -x_2J_2 & x_3J_2 - 2x_2J_3 \\ * & 0 & -x_3x_1 & x_1J_1 & x_1J_2 & 2x_1J_3 - x_3J_1 \\ * & * & 0 & 0 & 0 & -x_1J_2 + x_2J_1 \\ * & * & * & 0 & -J_1^2 - J_2^2 & -J_3J_2 \\ * & * & * & * & 0 & J_1J_3 \\ * & * & * & * & * & 0 \end{pmatrix} \tag{3.6}$$

has been found in [11]. It is easy to prove that P' has the same foliations by symplectic leaves as P at $C_2 = 0$.

3.2. Generic solution

Using quadratic in momenta ansatz (3.3) one got the generic solution depending on a pair of parameters μ and ν for all five cases of integrable systems (2.8).

Proposition 2. *For the integrable system (2.3) and (2.4), equations (1.2) and (1.3) have the following solution:*

$$P'_{\mu\nu} = \begin{pmatrix} 0 & \mathcal{X} & \mathcal{Y} + 4\,\mathrm{i}\alpha\mu p_\phi & 0 \\ * & 0 & \mathcal{Z} & 4\mathrm{i}\alpha\mu p_\phi \\ * & * & 0 & \mathcal{X}\frac{\mathrm{i}m(u)\,\mathrm{e}^{\mathrm{i}\phi}}{2\sqrt{1-u^2}} \\ * & * & * & 0 \end{pmatrix}, \tag{3.7}$$

where

$$\mathcal{X} = -2\mathrm{i}\mu(u+d) + \frac{\nu}{n(u)m(u)}, \qquad \mathrm{i} = \sqrt{-1},$$
$$\mathcal{Y} = \mathcal{X}\left[\mathrm{i}p_u + \left(\frac{3\alpha-1}{u+c} - \frac{u}{1-u^2}\right)p_\phi\right],$$
$$\mathcal{Z} = \frac{(-2\mathrm{i}\mu a^2(u+d)^{2\beta} - \nu)}{n(u)^2}\left[\left(\alpha - \frac{u(u+c)}{(1-u^2)}\right)p_u - \frac{\mathrm{i}(u+c)\mathrm{g}_\phi}{(1-u^2)}p_\phi\right],$$

Here g_ϕ *is a component of metric (3.2) and*

- $d = 0$ *at cases 1, 2, 4,*
- $d = A$ *at cases 3, 5.*

The complex conjugated bivector $\overline{P'_{\mu\nu}}$ *is another solution of the same equations (1.2), (1.3).*

In contrast with the particular solution (3.4) entries of this Poisson bivector are complex functions on initial variables and $P'_{\mu\nu}$ depends on parameter a instead of parameter b.

As above this Poisson bivector is compatible with P (3.1) and functions $H_{1,2}$ are in bi-involution with respect to the corresponding Poisson brackets. So, the phase space $T^*\mathcal{S}^2$ becomes the semisimple ωN manifold and the foliation defined by $H_{1,2}$ is separable in the Darboux–Nijenhuis variables, which are the eigenvalues of the recursion operator $N = P'P^{-1}$ [6].

Summing up, we have found particular and generic solutions of equations (1.2), (1.3) for five integrable systems (2.8) on the sphere with cubic integrals of motion. An important application of this result is the separation of variables for these systems.

4. Separation of variables

In this section we consider new separated variables and separated relations for the Goryachev system [2] and for the Goryachev–Chaplygin top [1] in detail.

4.1. The Goryachev system

In this case

$$\alpha = \beta = \frac{1}{3}, \qquad n = -ax_3^{1/3}, \qquad m = \frac{a}{x_3^{2/3}},$$

$$\ell = \frac{2a}{3x_3^{2/3}}, \qquad g = \frac{b}{x_3^{2/3}}, \qquad f = 1,$$

so that

$$H_1^{(2)} = J_1^2 + J_2^2 + \frac{4}{3}J_3^2 + \frac{ax_1}{x_3^{2/3}} + \frac{b}{x_3^{2/3}},$$

and

$$H_2^{(2)} = -\frac{2}{3}J_3\left(J_1^2 + J_2^2 + \frac{8}{9}J_3^2 + \frac{b}{x_3^{2/3}}\right) + ax_3^{1/3}J_1 - \frac{2ax_1J_3}{3x_3^{2/3}}.$$

If we put $\mu = 0$ and $\nu = -a^2$ in $P'_{\mu\nu}$ (3.7) one gets the following second bivector on $T^*\mathcal{S}^2$

$$P' = \begin{pmatrix} 0 & u^{1/3} & \mathrm{i}u^{1/3}p_u - \frac{u^{4/3}}{1-u^2}p_\phi & 0 \\ * & 0 & \frac{1-4u^2}{3u^{2/3}(1-u^2)}p_u + \frac{\mathrm{i}u^{1/3}(4-u^2)}{3(1-u^2)^2}p_\phi & 0 \\ * & * & 0 & \frac{\mathrm{i}a\,\mathrm{e}^{\mathrm{i}\phi}}{2u^{1/3}\sqrt{1-u^2}} \\ * & * & * & 0 \end{pmatrix}.$$

The variables of separation $q_{1,2}$ (the Darboux–Nijenhuis variables) are eigenvalues of the recursion operator $N = P'P^{-1}$, which are roots of the following polynomial:

$$\mathcal{A}(\lambda) = (\lambda - q_1)(\lambda - q_2) = \lambda^2 + u^{1/3}\left(\frac{up_\phi}{1-u^2} - \mathrm{i}p_u\right)\lambda - \frac{\mathrm{i}a\,\mathrm{e}^{\mathrm{i}\phi}}{2\sqrt{1-u^2}}. \quad (4.1)$$

In the initial $e^*(3)$ variables the second Poisson brackets look like

$$\{x_i, x_j\}' = \varepsilon_{ijk}x_k x_3^{1/3}, \qquad \{x_j, J_3\}' = 0,$$

$$\{x_1, J_1\}' = \frac{x_2 J_1}{3x_3^{2/3}} - \frac{x_3^{4/3} J_2}{x_1 + \mathrm{i}x_2} + \frac{4x_3^{1/3} x_2 J_3}{3(x_1 + \mathrm{i}x_2)},$$

$$\{x_2, J_2\}' = \frac{\mathrm{i}x_3^{4/3} J_1}{x_1 + \mathrm{i}x_2} - \frac{x_1 J_2}{3x_3^{2/3}} - \frac{4\mathrm{i}x_3^{1/3} x_1 J_3}{3(x_1 + \mathrm{i}x_2)},$$

$$\{x_1, J_2\}' = \frac{\mathrm{i}\left(x_2^2 - \mathrm{i}x_1x_2 - 3x_3^2\right)J_2}{3x_3^{2/3}(x_1 + \mathrm{i}x_2)} + \frac{4\mathrm{i}x_3^{1/3} x_2 J_3}{3(x_1 + \mathrm{i}x_2)},$$

$$\{x_2, J_2\}' = -\frac{\left(x_1^2 + \mathrm{i}x_1x_2 - 3x_3^2\right)J_1}{3x_3^{2/3}(x_1 + \mathrm{i}x_2)} - \frac{4x_3^{1/3} x_1 J_3}{3(x_1 + \mathrm{i}x_2)},$$

$$\{x_3, J_1\}' = -\frac{x_3^{1/3}(J_1x_2 - x_1J_2)}{x_1 + \mathrm{i}x_2}, \qquad \{x_3, J_2\}' = -\frac{\mathrm{i}x_3^{1/3}(J_1x_2 - x_1J_2)}{x_1 + \mathrm{i}x_2},$$

$$\{J_i, J_j\}' = \frac{-a\varepsilon_{ijk}x_k}{2x_3^{1/3}(x_1 + \mathrm{i}x_2)} + \delta_{i1}\delta_{j2}\frac{\mathrm{i}(J_1x_2 - x_1J_2)(J_1 + \mathrm{i}J_2)}{3x_3^{2/3}(x_1 + \mathrm{i}x_2)}.$$

The corresponding Poisson bivector P' is compatible with canonical Poisson bivector on $e^*(3)$ and satisfies to equations (1.4), so that P' has the same foliations by symplectic leaves as P at $C_2 = 0$.

According to [6] the bi-involutivity of integrals of motion

$$\left\{H_1^{(2)}, H_2^{(2)}\right\} = \left\{H_1^{(2)}, H_2^{(2)}\right\}' = 0$$

is equivalent to the existence of the non-degenerate control matrix F such that

$$P'\,\mathrm{d}H_i^{(2)} = P\sum_{j=1}^{2} F_{ij}\,\mathrm{d}H_j^{(2)}, \qquad i = 1, 2. \tag{4.2}$$

In our case the control matrix F reads as

$$F = \begin{pmatrix} -\dfrac{x_3^{1/3}(J_1 + \mathrm{i}J_2)}{x_1 + \mathrm{i}x_2} + \dfrac{2J_3}{3x_3^{2/3}} & \dfrac{1}{x_3^{2/3}} \\ \dfrac{-ax_3^{2/3}}{2(x_1 + \mathrm{i}x_2)} + \dfrac{2x_3^{1/3}(J_1 + \mathrm{i}J_2)J_3}{3(x_1 + \mathrm{i}x_2)} - \dfrac{4J_3^2}{9x_3^{2/3}} & \dfrac{-2J_3}{3x_3^{2/3}} \end{pmatrix}.$$

The Darboux–Nijenhuis variables $q_{1,2}$ are simultaneously eigenvalues of the recursion operator and eigenvalues the control matrix. In our case they are the roots of the following polynomial:

$$\mathcal{A}(\lambda) = \det(F - \lambda\mathrm{I}) = \lambda^2 + \frac{x_3^{1/3}(J_1 + \mathrm{i}J_2)}{x_1 + \mathrm{i}x_2}\lambda + \frac{a}{2(x_1 + \mathrm{i}x_2)}.$$

The left eigenvectors of F, if suitably normalized, form the Stäckel matrix S, which enters into a pair of the separated relations

$$\sum_{j=1}^{2} S_{ij}(q_i, p_i)H_j^{(2)} - U_i(q_i, p_i) = 0, \qquad i = 1, 2. \tag{4.3}$$

Here U_i are the Stäckel potentials and $p_{1,2}$ are variables conjugated to $q_{1,2}$

$$\{q_i, p_j\} = \delta_{ij}, \qquad \{q_i, p_j\}' = \delta_{ij}q_i, \qquad \{q_i, q_j\} = \{q_i, q_j\}' = \{p_i, p_j\} = \{p_i, p_j\}' = 0.$$

Unfortunately, construction of the variables $p_{1,2}$ is the non-algorithmic procedure, which depends on the fortune and skillfulness [6]. In our case we can observe that

$$q_1 + q_2 = \mathrm{i}u^{1/3}p_u - \frac{u^{4/3}p_\phi}{1-u^2}, \qquad \text{and} \qquad q_1q_2 = -\frac{\mathrm{i}a\,\mathrm{e}^{\mathrm{i}\phi}}{2\sqrt{1-u^2}},$$

so that

$$\{p_\phi, q_1+q_2\} = 0, \qquad \{p_\phi, q_1q_2\} = -\mathrm{i}q_1q_2, \qquad \{u, q_1+q_2\} = \mathrm{i}u^{1/3}, \qquad \{u, q_1q_2\} = 0.$$

Integrating these equations with respect to $p_\phi(q,p)$ and $u(q,p)$ we can easily get the following expressions for these functions

$$p_\phi = \frac{\mathrm{i}q_1q_2(p_2-p_1)}{q_1-q_2}, \qquad \text{and} \qquad u = \left(\frac{-2\mathrm{i}(q_1p_1 - q_2p_2)}{3(q_1-q_2)}\right)^{3/2},$$

which yield the necessary definitions of the momenta

$$p_i = \mathcal{B}(\lambda = q_i), \qquad i = 1, 2,$$

where

$$\mathcal{B}(\lambda) = \mathrm{i}\left(\frac{3u^{2/3}}{2} - \frac{p_\phi}{\lambda}\right) = \mathrm{i}\left(\frac{3x_3^{2/3}}{2} + \frac{J_3}{\lambda}\right).$$

Using the following relations for the second Poisson bracket

$$\{p_\phi, q_1+q_2\}' = q_1q_2, \qquad \text{and} \qquad \{p_\phi, q_1q_2\}' = 0,$$

it is easy to prove that $\{q_i, p_j\}' = \delta_{ij}q_i$.

In the separated variables the Stäckel matrix S is equal to

$$S = \begin{pmatrix} 1 & 1 \\ \frac{3\mathrm{i}}{2q_1p_1} & \frac{3\mathrm{i}}{2q_2p_2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{-1}{2\mathrm{i}\alpha q_1p_1} & \frac{-1}{2\mathrm{i}\alpha q_2p_2} \end{pmatrix},$$

whereas integrals of motion have the generalized Stäckel form at the second case (2.8)

$$H_i^{(2)} = \sum_{j=1}^{2} S_{ij}^{-1}\left(\frac{\left(4p_j^2 + 6\mathrm{i}p_j - 9\right)(3\mathrm{i} - 2p_j)q_j^2}{18p_j} - \frac{3\mathrm{i}a^2}{8q_j^2p_j} + \frac{3\mathrm{i}b}{2p_j}\right), \qquad i = 1, 2.$$

The corresponding separated relations (4.3) define two copies of the following algebraic curve:

$$\mathcal{C}: \quad \mu^3 - H_1^{(2)}\mu + \lambda^3 = b\lambda - H_2^{(2)} - \frac{a^2}{4\lambda}, \qquad \mu = 2\mathrm{i}\alpha q_i p_i, \qquad \lambda = q_i. \tag{4.4}$$

In contrast with the usual Stäckel systems it is a non-hyperelliptic curve, which is a trigonal algebraic curve. It is interesting that one formally gets the real algebraic curve. But we have to bear firmly in mind that our separated variables $q_{1,2}$ and $p_{1,2}$ are complex functions on initial physical variables.

If we put $\nu = 0$ and $\mu = 1$ in $P'_{\mu\nu}$ (3.7) one gets another Poisson bivector on $T^*\mathcal{S}^2$

$$P' = \begin{pmatrix} 0 & u & \mathrm{i}up_u - \frac{2+u^2}{3(1-u^2)}p_\phi & 0 \\ * & 0 & \frac{1-4u^2}{3(1-u^2)}p_u + \frac{\mathrm{i}u(4-u^2)}{3(1-u^2)^2}p_\phi & -\frac{2}{3}p_\phi \\ * & * & 0 & \frac{\mathrm{i}au^{1/3}\,\mathrm{e}^{\mathrm{i}\phi}}{2\sqrt{1-u^2}} \\ * & * & * & 0 \end{pmatrix}.$$

In this case the Darboux–Nijenhuis coordinates are roots of the following polynomial:

$$\mathcal{A}(\lambda) = \lambda^2 - \left(\mathrm{i}up_u - \frac{4-u^2}{3(1-u^2)}p_\phi\right) - \frac{\mathrm{i}au^{4/3}\,\mathrm{e}^{\mathrm{i}\phi}}{2\sqrt{1-u^2}} - \frac{2\mathrm{i}u}{3}p_up_\phi + \frac{2(2+u^2)}{9(1-u^2)}p_\phi^2.$$

In terms of the previous separated variables this polynomial looks like

$$\mathcal{A}(\lambda) = \left(\lambda + \frac{2\mathrm{i}}{3} q_1 p_1\right)\left(\lambda + \frac{2\mathrm{i}}{3} q_2 p_2\right).$$

Now it is easy to prove that in the separated variables (q, p) bivector $P'_{\mu\nu}$ (3.7) looks like

$$P'_{\mu\nu} = -2\mathrm{i}\alpha\mu \begin{pmatrix} 0 & 0 & q_1p_1 & 0 \\ 0 & 0 & 0 & q_2p_2 \\ -q_1p_1 & 0 & 0 & 0 \\ 0 & -q_2p_2 & 0 & 0 \end{pmatrix} - \frac{\nu}{a^2}\begin{pmatrix} 0 & 0 & q_1 & 0 \\ 0 & 0 & 0 & q_2 \\ -q_1 & 0 & 0 & 0 \\ 0 & -q_2 & 0 & 0 \end{pmatrix}. \qquad (4.5)$$

For the remaining four systems bivector $P_{\mu,\nu}$ (3.7) has the similar form in the Darboux–Nijenhuis variables (q, p).

4.2. *The generalized Goryachev–Chaplygin top*

Now we briefly discuss a new separation of variables for the generalized Goryachev–Chaplygin top [1]. Recall that in this case the Hamiltonian reads as

$$H_1^{(1)} = J_1^2 + J_2^2 + 4J_3^2 - ax_1 + \frac{b}{x_3^2}.$$

The new separated coordinates are roots of the polynomial

$$\begin{aligned}\mathcal{A}(\lambda) &= \lambda^2 + a^2\left(\frac{\mathrm{i}p_u}{mn} + \frac{3\alpha p_\phi}{mn(u+c)} + \frac{p_\phi(uc+1)}{mn(u+c)(u^2-1)}\right)\lambda + \frac{\mathrm{i}a^4\mathrm{e}^{\mathrm{i}\phi}}{2mn^2\sqrt{1-u^2}} \\ &= \lambda^2 - \left(\frac{\mathrm{i}p_u}{u} + \frac{3p_\phi}{u^2} + \frac{p_\phi}{u^2(u^2-1)}\right)\lambda + \frac{\mathrm{i}a\,\mathrm{e}^{\mathrm{i}\phi}}{2u^2\sqrt{1-u^2}},\end{aligned}$$

whereas the conjugated momenta are equal to

$$p_i = \mathcal{B}(\lambda = q_i), \qquad \text{where} \quad \mathcal{B}(\lambda) = \mathrm{i}\left(\frac{u^2}{2} - \frac{p_\phi}{\lambda}\right).$$

These separated variables lie on two copies of the following algebraic curve:

$$\mathcal{C}: \quad \mu\left(\mu^2 + \mu\lambda - H_1^{(1)}\right) = b\lambda - H_2^{(1)} - \frac{a^2}{4\lambda}, \qquad \mu = 2\mathrm{i}\alpha q_i p_i, \qquad \lambda = q_i. \qquad (4.6)$$

As for the Goryachev model these separated variables $q_{1,2}$ and $p_{1,2}$ are complex functions on initial physical variables, which lie on the non-hyperelliptic algebraic curve.

Of course, we can repeat similar calculations for the remaining systems and prove that they are related with different trigonal curves as well. As an example, for the Dullin–Matveev system [4] the equations of motion are linearized on the following algebraic curve:

$$\mathcal{C}: \quad \mu\left(\mu^2 - \lambda^2 - H_1^{(4)}\right) = b\lambda - H_2^{(4)} - \frac{a^2}{4\lambda}, \qquad \mu = 2\mathrm{i}\alpha q_i p_i, \qquad \lambda = q_i, \qquad (4.7)$$

if $c = 0$ in the Hamiltonian $H_1^{(4)}$ (2.8).

It will be interesting to get solutions of the equations of motion in terms of the Abelian functions for trigonal curves, as an example see [5]. The other open question is the construction of the Lax matrices associated with separated variables on trigonal curves.

4.3. The Jacobi method

The Jacobi method consists of the construction of the integrable system starting with some known separated variables and arbitrary separated relations. The method was originally formulated by Jacobi when he invented elliptic coordinates and successfully applied them to solve several important mechanical problems, such as the problem of geodesic motion on an ellipsoid, and the problem of planar motion in a force field of two attracting centers. In [7] Jacobi himself wrote: 'The main difficulty in integrating a given differential equation lies in introducing convenient variables, which there is no rule for finding. Therefore, we must travel the reverse path and after finding some notable substitution, look for problems to which it can be successfully applied'.

In our case this notable substitution is described by the following

Proposition 3. *At any a, c and α transformation*

$$\begin{gathered} u=\left(\frac{-2\mathrm{i}\alpha(p_1q_1-q_2p_2)}{q_1-q_2}\right)^{1/2\alpha}-c, \qquad p_\phi=\frac{-\mathrm{i}q_1q_2(p_1-p_2)}{q_1-q_2},\\ \phi=-\frac{\mathrm{i}}{2}\left(2\ln(u+c)(3\alpha-1)+\ln\left(\frac{4q_1^2q_2^2(u^2-1)}{a^2}\right)\right), \\ p_u=\left(\frac{3\mathrm{i}\alpha}{u+c}+\frac{\mathrm{i}(uc+1)}{(u+c)(u^2-1)}\right)p_\phi-\mathrm{i}(q_1+q_2)(u+c)^{2\alpha-1} \end{gathered} \tag{4.8}$$

is a canonical transformation.

Now in order to get some integrable system on $T^*\mathcal{S}^2$ we could take two copies of any algebraic curve $\mathcal{C}$ defined by equation $\Phi(\mu,\lambda,H_1,H_2)=0$ and solve the corresponding separated relations

$$\Phi(\mu_i,\lambda_i,H_1,H_2)=0, \qquad \mu_i=2\mathrm{i}\alpha q_ip_i, \qquad \lambda_i=q_i$$

with respect to integrals of motion H_1 and H_2, which will be in bi-involution [14]. As above the main problem is that change of variables (4.8) is the transformation over the complex field $\mathbb{C}$ and if we want to get real functions $H_{1,2}$ on the initial variables on $T^*\mathcal{S}^2$ we have to start with the very special algebraic curves $\mathcal{C}$, for instance with (4.4), (4.6) or (4.7).

Example. Let us consider the following deformation of the algebraic curve (4.6):

$$\widetilde{\mathcal{C}}: \quad \mu(\mu^2+\mu\lambda-H_1)+\rho\left(\frac{\rho}{4}+\mu\right)(\lambda+\mu)=b\lambda-H_2-\frac{a^2}{4\lambda}. \tag{4.9}$$

Solving the corresponding separated relations with respect to $H_{1,2}$ one gets integrals of motion in the bi-involution. After canonical transformation (4.8) at $c=0$ and $\alpha=1$ we obtain the following complex Hamiltonian:

$$\begin{aligned} H_1=(1-u^2)p_u^2+\frac{(4-3u^2)p_\phi^2}{1-u^2}-a\sin(\phi)\sqrt{1-u^2}+\frac{b}{u^2}\\ +\rho p_\phi-\frac{\rho^2(1-u^2)}{4u^2}+\frac{\mathrm{i}\rho p_u(1-u^2)}{u}, \end{aligned}$$

which after an obvious additional shift of momenta (canonical transformation)

$$p_u\to\tilde{p}_u=p_u-\frac{\mathrm{i}\rho}{2u}$$

becomes the real Hamiltonian for the Goryachev–Chaplygin gyrostat

$$H_1=H_1^{(1)}+\rho p_\phi=J_1^2+J_2^2+4J_3^2-\rho J_3-ax_1+\frac{b}{x_3^2}.$$

More complicated deformations of the algebraic curve (4.6), such as

$$\mu(\mu^2+\mu\lambda-H_1)+c_1\mu\lambda+c_2\lambda^2+c_3\mu^2=b\lambda-H_2-\frac{a^2}{4\lambda},$$

and the corresponding additional shifts of the momenta p_u lead to another generalizations of the Goryachev–Chaplygin top, which was obtained in [16].

Of course, we can get similar deformations for the remaining four systems in (2.8) as well.

5. The Kowalevski top on the sphere

The proposed approach may be useful for investigations of the other integrable systems on the sphere too. As an example for the systems with the fourth-order additional integrals of motion we could apply the same anzats (3.3) but at $N = 3$. In this case entries of the vector field X^j will be generic third-order polynomials in momenta p_u, p_ϕ with coefficients $y^j_{km}(u,\phi)$ depending on variables u and ϕ.

Example. If we substitute the Hamiltonian of the Kowalevski top on the unit sphere

$$\begin{aligned} H_1 &= J_1^2+J_2^2+2J_3^2-2bx_1\\ &=(1-u^2)p_u^2+\frac{(2-u^2)p_\phi^2}{1-u^2}-2b\sqrt{1-u^2}\sin\phi, \end{aligned} \tag{5.1}$$

and the second integral of motion

$$H_2=\left(J_1^2+J_2^2\right)^2+4\left(x_1\left(J_1^2-J_2^2\right)+2x_2J_1J_2\right)b+4\left(x_1^2+x_2^2\right)b^2$$

into equations (1.2), (1.3), then directly solving these equations one gets the following second Poisson bracket:

$$\begin{aligned} &\{u,\phi\}'=up_\phi-\mathrm{i}(1-u^2)p_u, \qquad \{u,p_u\}'=\frac{(1-u^2)p_u^2}{2}-\frac{u^2p_\phi^2}{2(1-u^2)},\\ &\{u,p_\phi\}'=(1-u^2)p_up_\phi, \qquad \{p_u,p_\phi\}'=\left(p_u^2-\frac{p_\phi^2}{(1-u^2)^2}\right)up_\phi,\\ &\{\phi,p_u\}'=\mathrm{i}u\left(p_u^2-\frac{p_\phi^2}{(1-u^2)^2}\right)+\frac{(1-2u^2)p_up_\phi}{1-u^2}-\frac{bu\,\mathrm{e}^{\mathrm{i}\phi}}{\sqrt{1-u^2}},\\ &\{\phi,p_\phi\}'=\frac{(1-u^2)p_u^2}{2}+\frac{(4-u^2)p_\phi^2}{2(1-u^2)}+\mathrm{i}b\sqrt{1-u^2}\,\mathrm{e}^{\mathrm{i}\phi}, \end{aligned} \tag{5.2}$$

which is compatible with canonical ones.

The separated variables for the Kowalevski top on the sphere appear as predicted by the general theory as eigenvalues of the corresponding recursion operator. In fact these separated variables coincide with the separated variables which have been introduced in the framework of the r-matrix formalism [8]. In this case our solution (5.2) of equations (1.2), (1.3) may be extracted from the second Poisson structure for the reflection equation algebra [15].

As above, using these separated variables we could get different generalizations of the Kowalevski top on the sphere in the framework of the Jacobi method.

6. Conclusion

We found the two-parametric Poisson bivector (3.7), which is compatible with the canonical Poisson bivector on cotangent bundle $T^*\mathcal{S}^2$ of the two-dimensional sphere. The quadratic and cubic integrals of motion (2.3), (2.4) for the five integrable systems on the sphere are in bi-involution with respect to the corresponding Poisson brackets.

The eigenvalues of the corresponding recursion operator are the separated coordinates. For the Goryachev system and Goryachev–Chaplygin top we give an explicit formulae for these separated variables and the corresponding separated relations.

Acknowledgment

The research was partially supported by the RFBR grant.

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